Simple math for a complex world: Random walks in biology and finance

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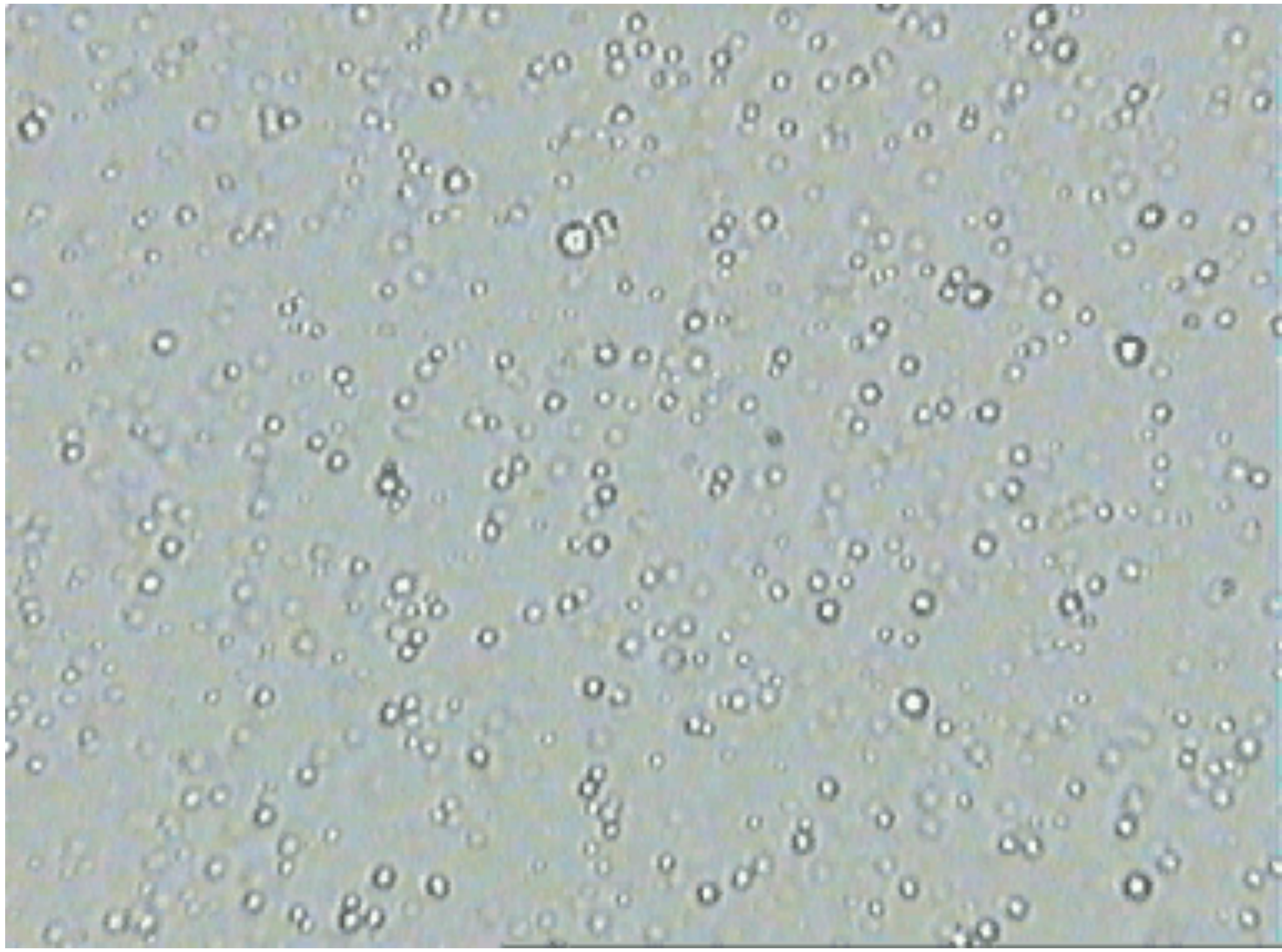
2007.10.31

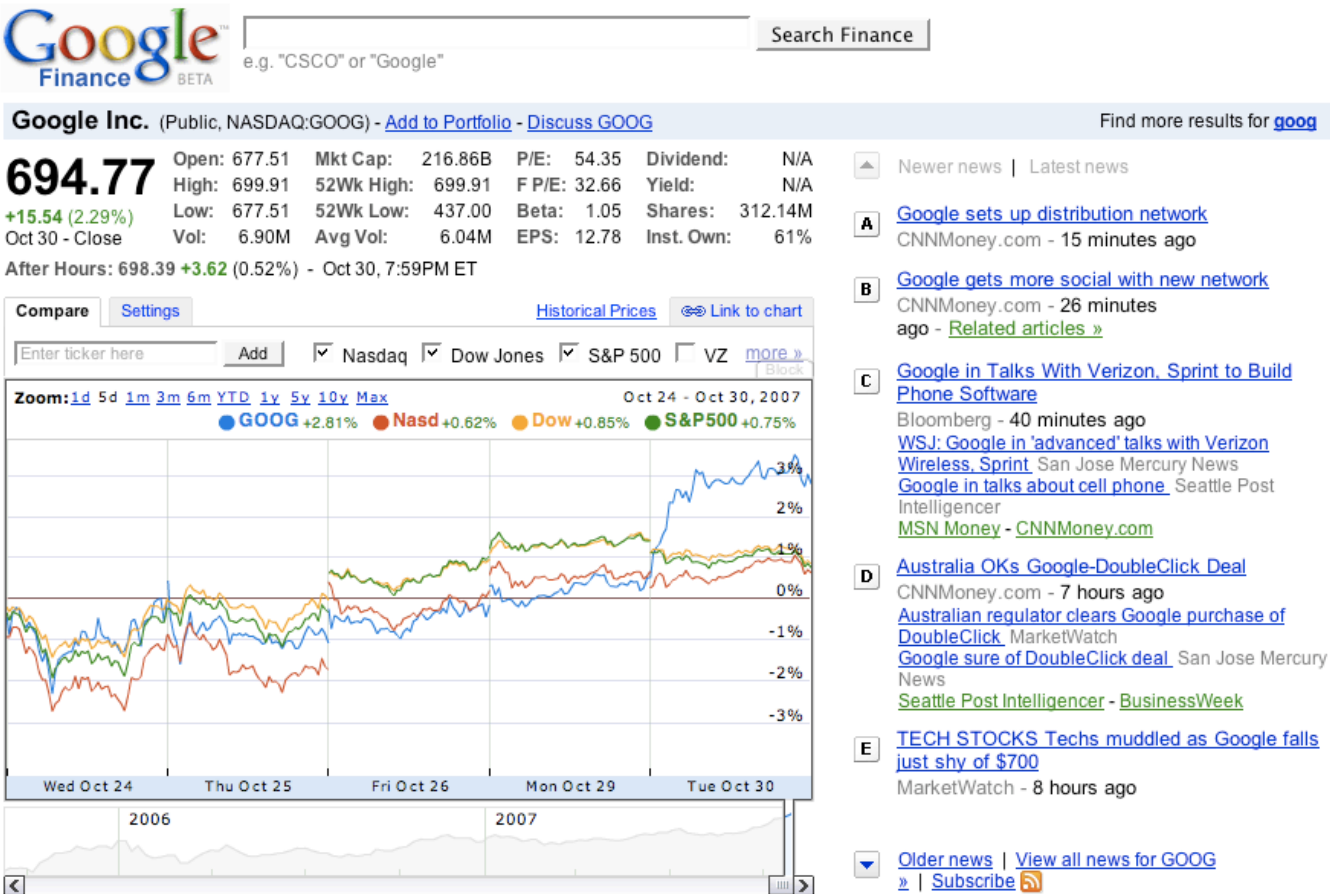
Outline

- Complex systems
- The atomic hypothesis and Brownian motion
- Mathematics of random walks
- Random walks in biology
- Random walks in finance
- Conclusions

Complex systems

It's difficult to model complex systems, even when we (think we) know the fundamental principles by which they are governed.





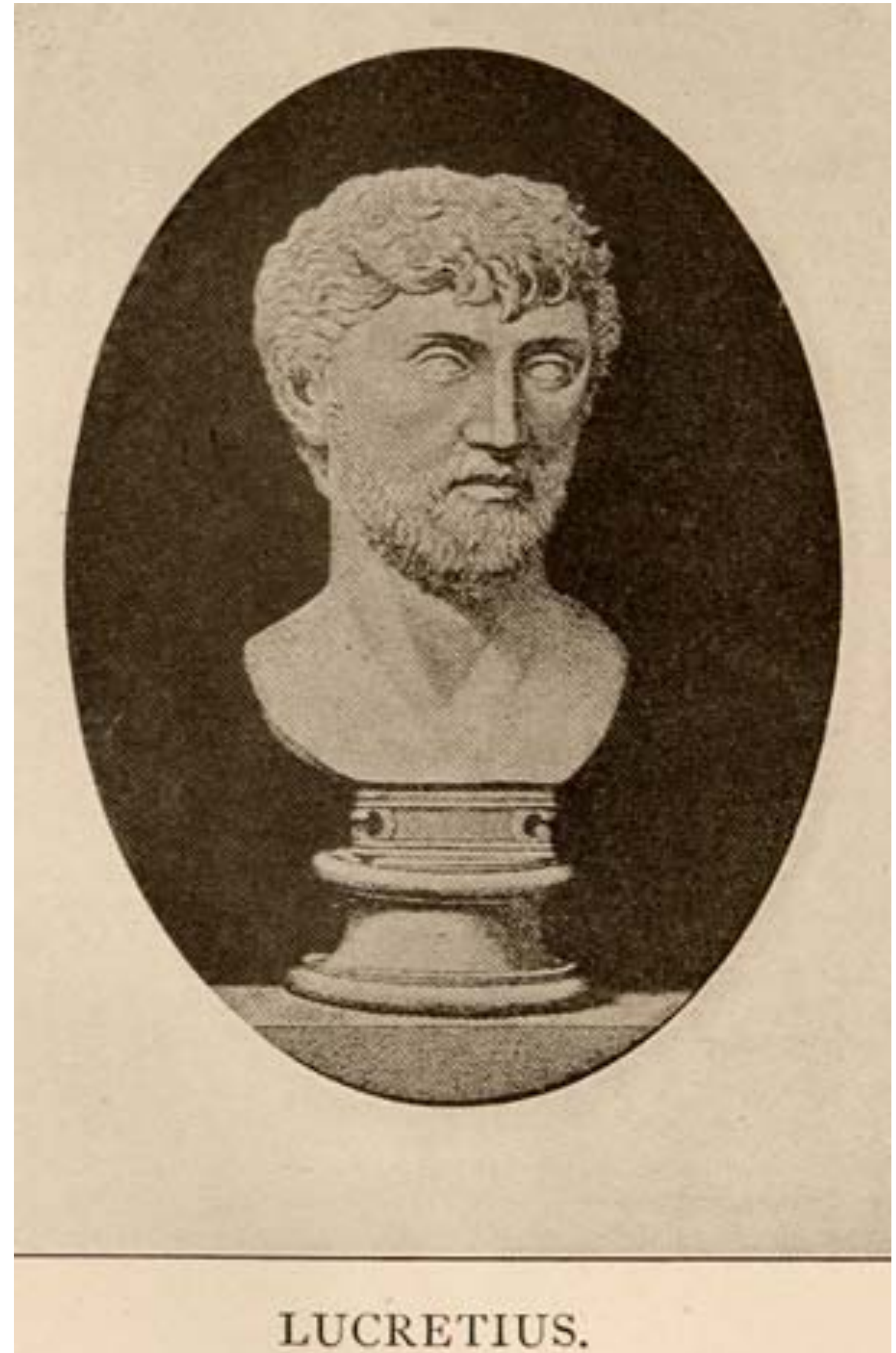
Complex systems

We would also like to model systems for which fundamental principles are unknown (or may not exist)

Complex systems

- **So how can we say anything meaningful, let alone *quantitative*, about these systems?**
  - Collective behavior of many random things can be quantified, even when behavior of individual things is random or unpredictable
  - Relatively simple mathematics; mostly probability and statistics
  - Many seemingly disparate systems united by mathematics

The atomic hypothesis



The atomic hypothesis

"Observe what happens when sunbeams are admitted into a building and shed light on its shadowy places. You will see a multitude of tiny particles mingling in a multitude of ways... their dancing is an actual indication of underlying movements of matter that are hidden from our sight... It originates with the atoms which move of themselves [i.e. spontaneously]. Then those small compound bodies that are least removed from the impetus of the atoms are set in motion by the impact of their invisible blows and in turn cannon against slightly larger bodies. So the movement mounts up from the atoms and gradually emerges to the level of our senses, so that those bodies are in motion that we see in sunbeams, moved by blows that remain invisible." - *On the Nature of Things,* Lucretius, 60 B.C.

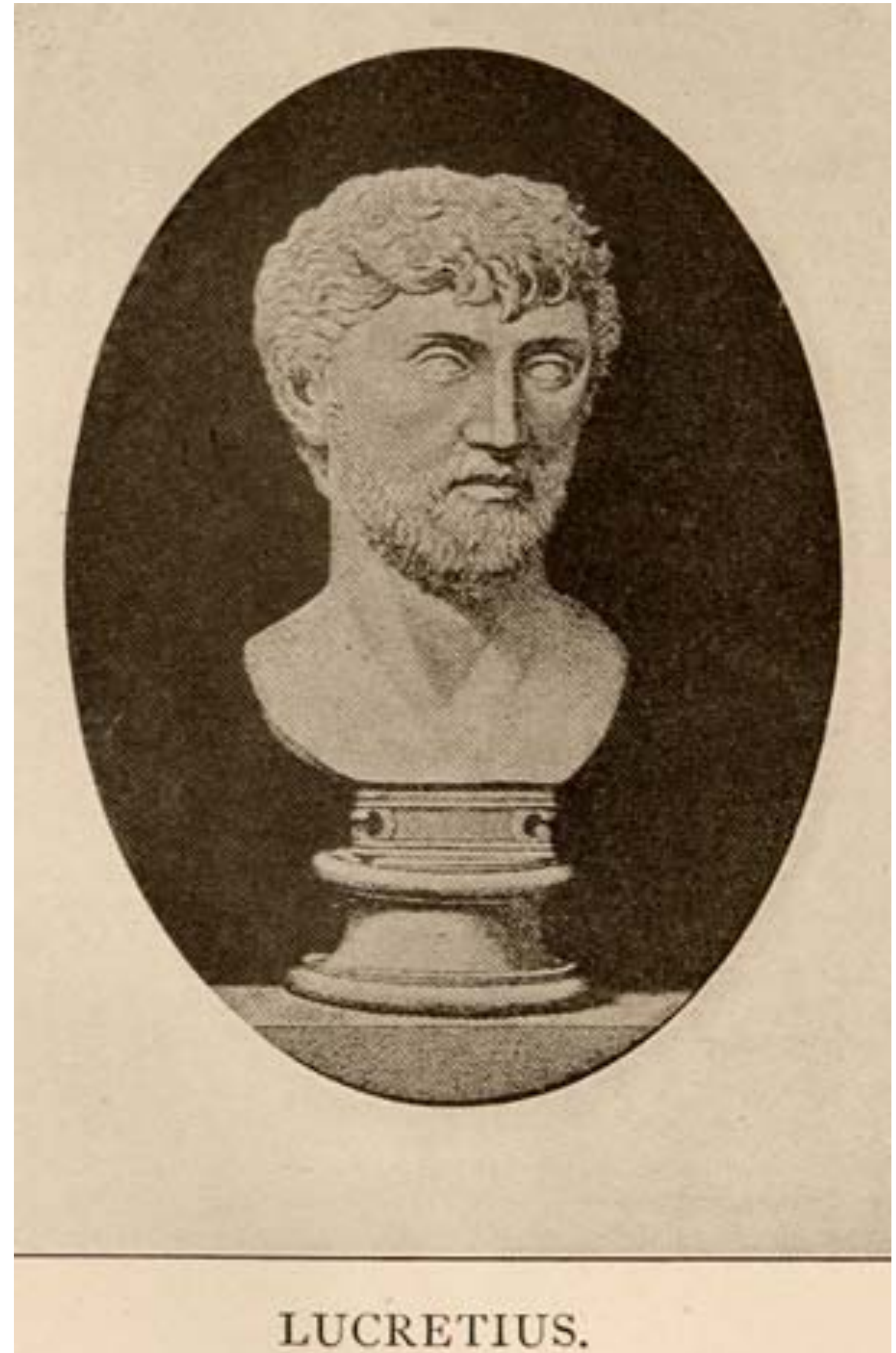
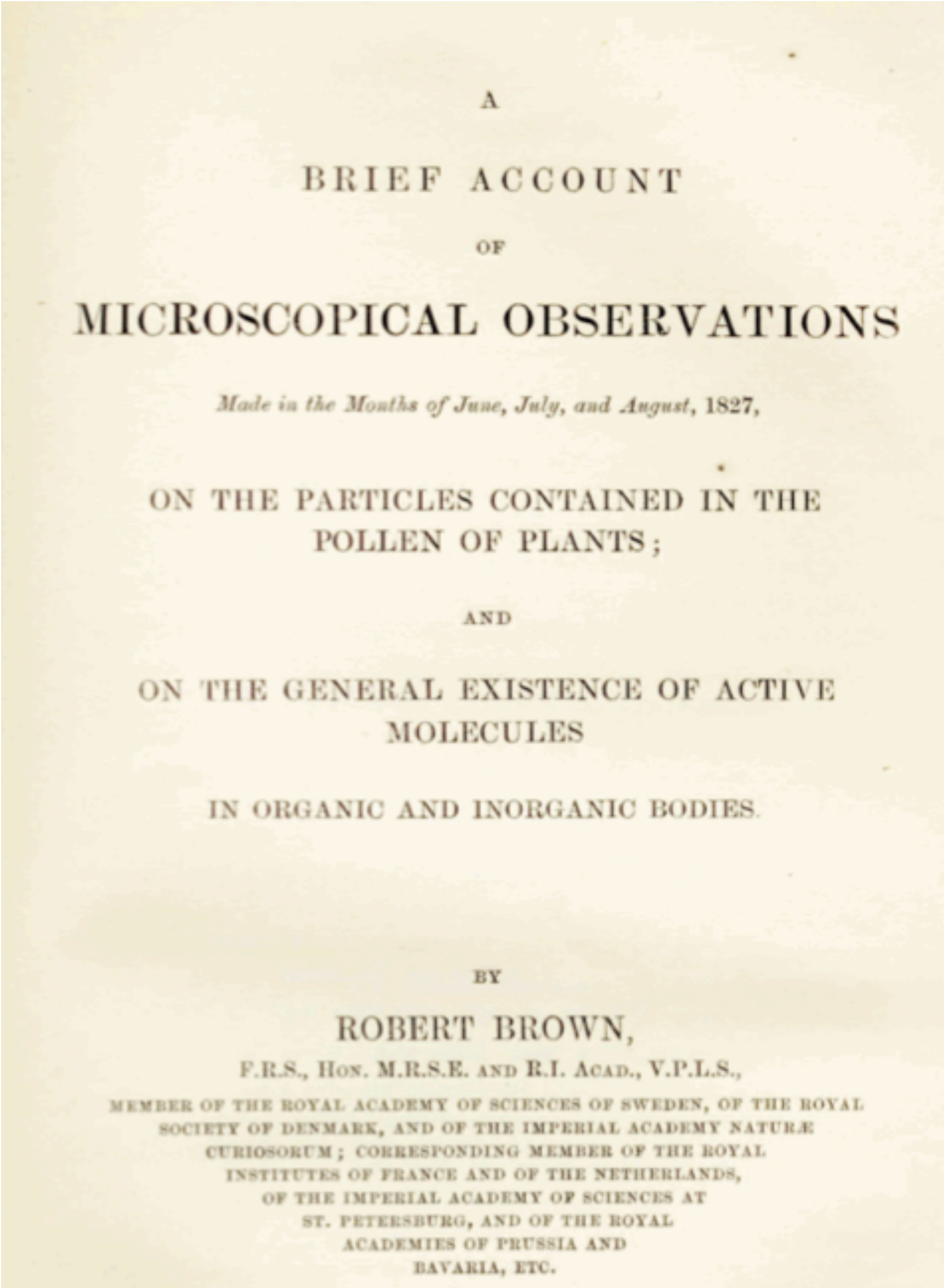


Photo from *Ben Franklin Stilled the Waves*, Tanford 1989

The size of a molecule

Ben Franklin finds that a teaspoon of oil covers the surface of a pond, half an acre in surface area; Lord Rayleigh calculates molecular size



A

BRIEF ACCOUNT

OF

MICROSCOPICAL OBSERVATIONS

Made in the Months of June, July, and August, 1827,

ON THE PARTICLES CONTAINED IN THE POLLEN OF PLANTS;

AND

ON THE GENERAL EXISTENCE OF ACTIVE MOLECULES

IN ORGANIC AND INORGANIC BODIES.

BY

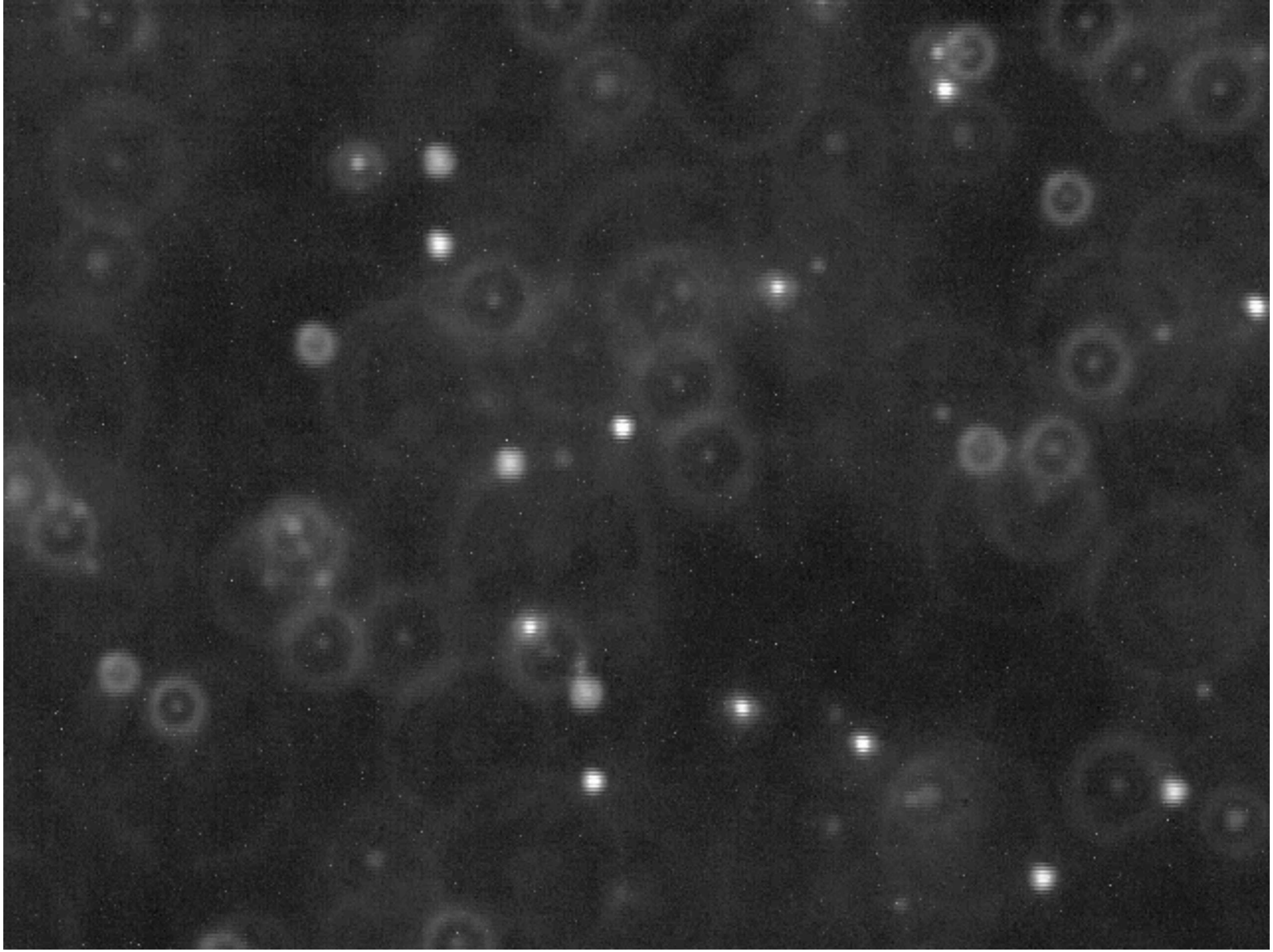
ROBERT BROWN,

F.R.S., HON. M.R.S.E. AND R.I. ACAD., V.P.L.S.,

MEMBER OF THE ROYAL ACADEMY OF SCIENCES OF SWEDEN, OF THE ROYAL SOCIETY OF DENMARK, AND OF THE IMPERIAL ACADEMY NATURÆ CURIOSORUM; CORRESPONDING MEMBER OF THE ROYAL INSTITUTES OF FRANCE AND OF THE NETHERLANDS, OF THE IMPERIAL ACADEMY OF SCIENCES AT ST. PETERSBURG, AND OF THE ROYAL ACADEMIES OF PRUSSIA AND BAVARIA, ETC.

http://www.brianjford.com/wbbrowna.htm

Robert Brown observes the “incessant dance” of pollen grains in 1827

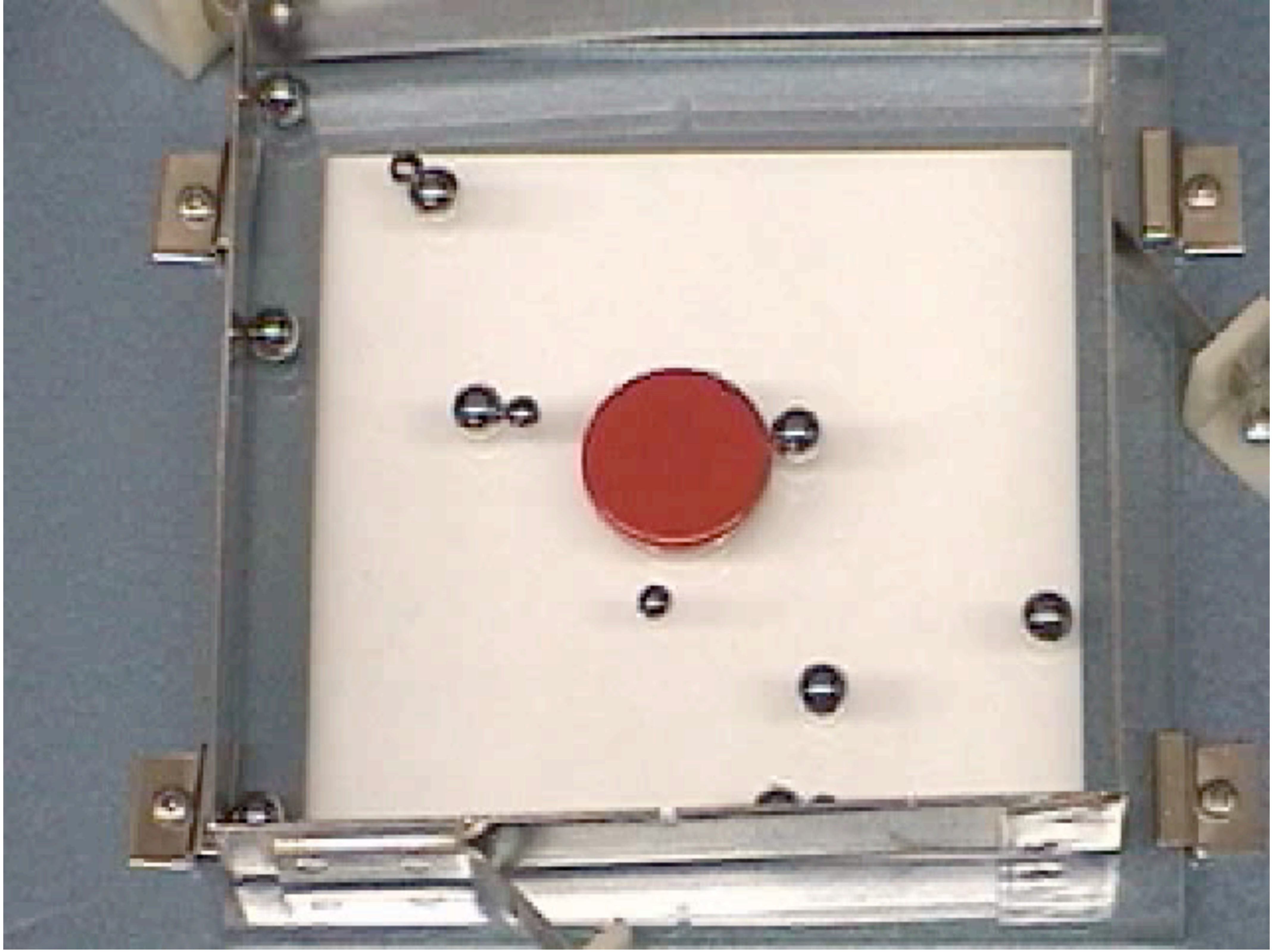


Brownian motion

Decades later, postulate that observed motion is due to frequent (10^{12} / sec) collisions with small molecules (10^{-3} * size pollen)

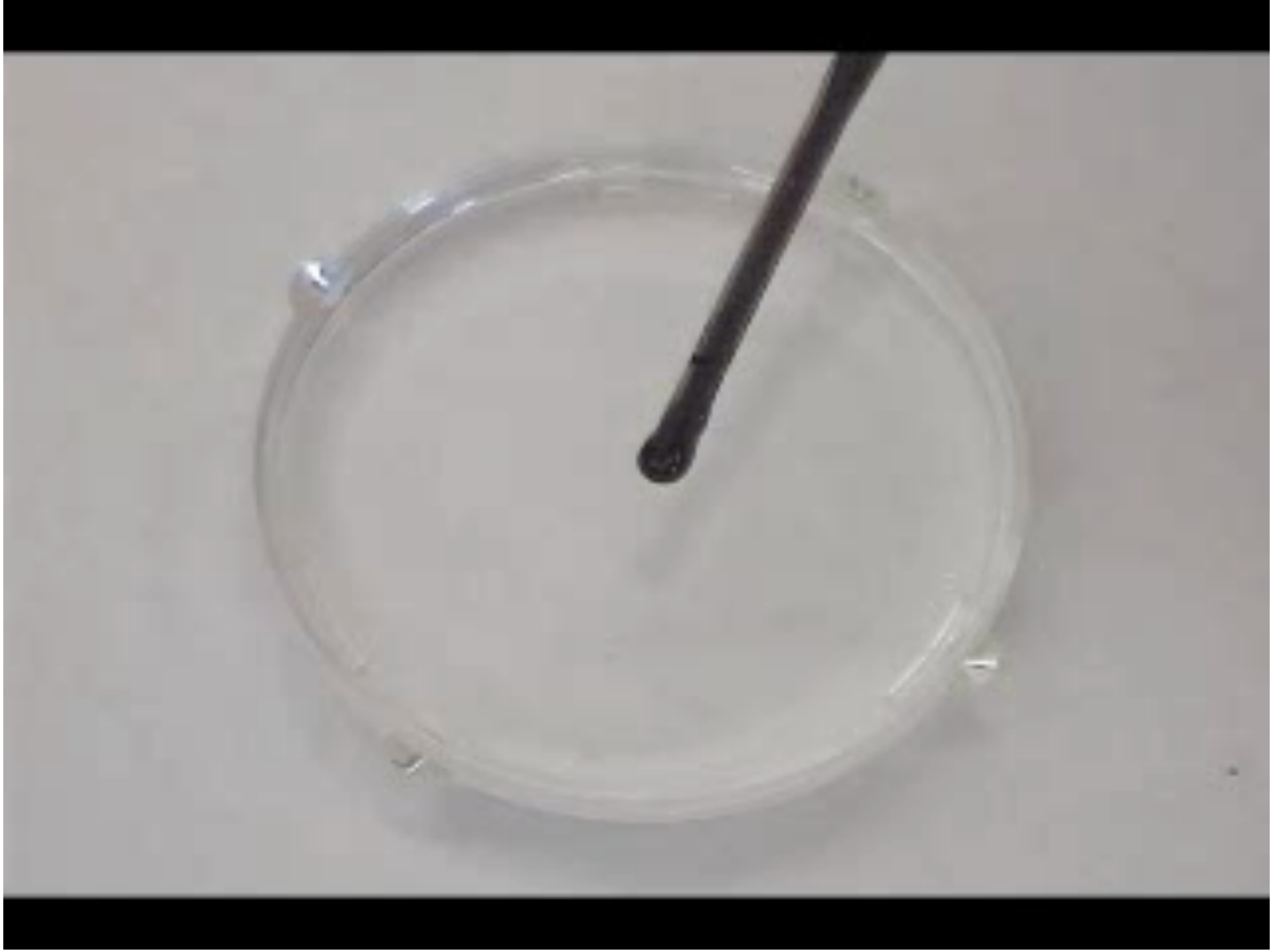
http://www.cs.princeton.edu/courses/archive/fall05/cos226/assignments/atomic.html

Bachelier and Einstein to the rescue: Brownian motion is quantified as a “random walk” and falsifiable experiments are formed; results support atomic hypothesis



Brownian motion

We observe rare and infrequent large displacements; see also: http://nanoatlas.ifs.hr/images/brownian_motion_my.swf

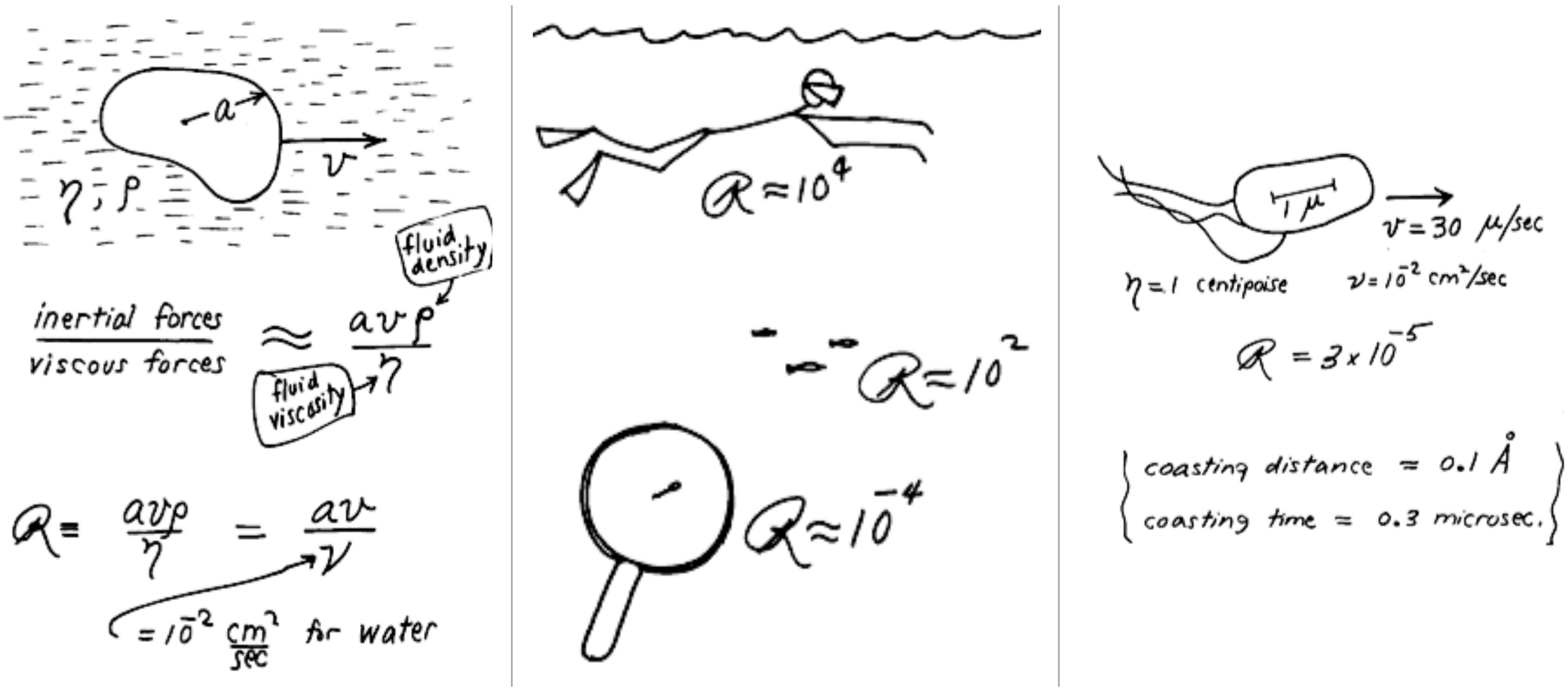


Diffusion

Observe many particles and we see average or *expected* behavior

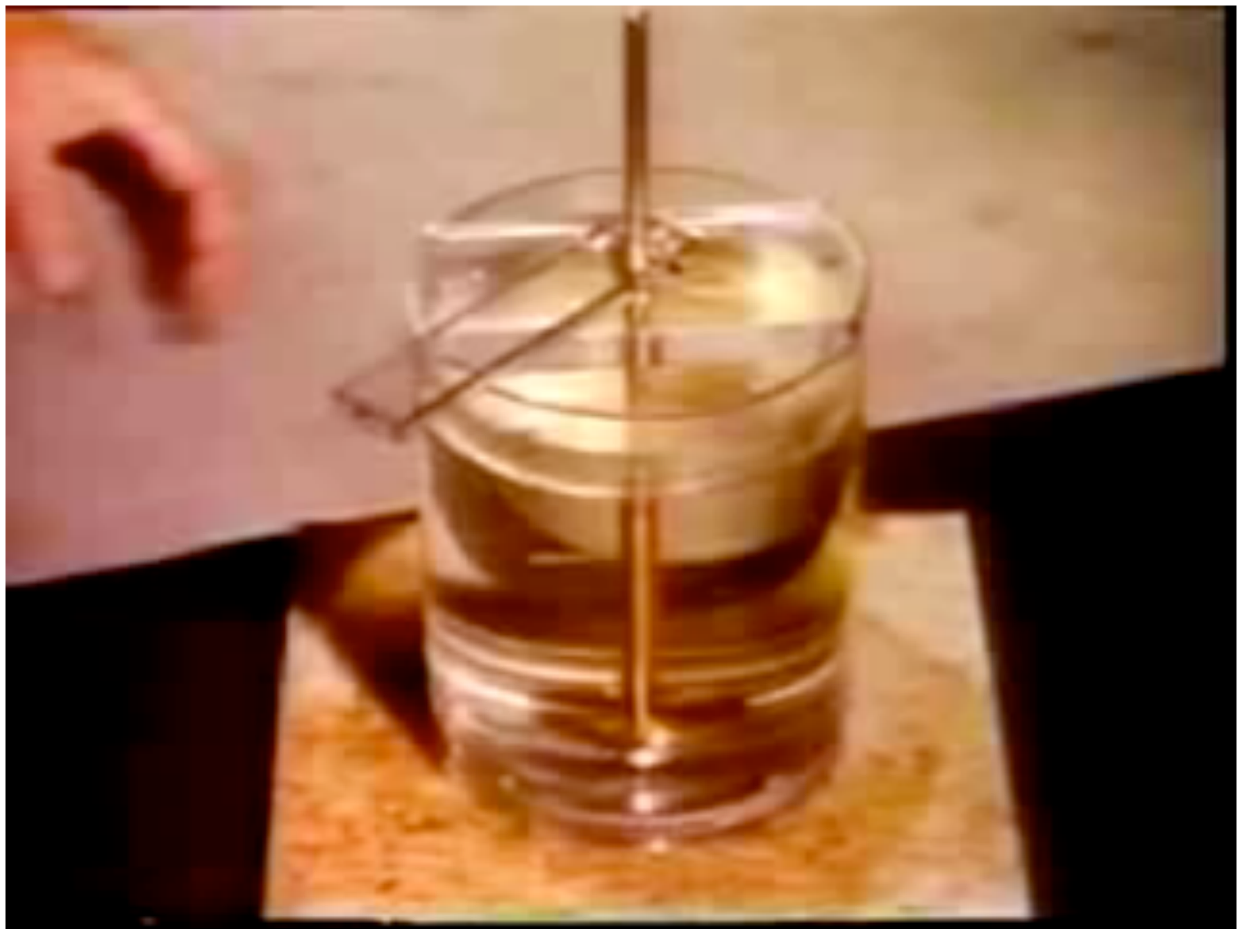
Life at low Reynolds number

E. M. Purcell
Lyman Laboratory, Harvard University, Cambridge, Massachusetts 02138
(Received 12 June 1976)



Reynolds number

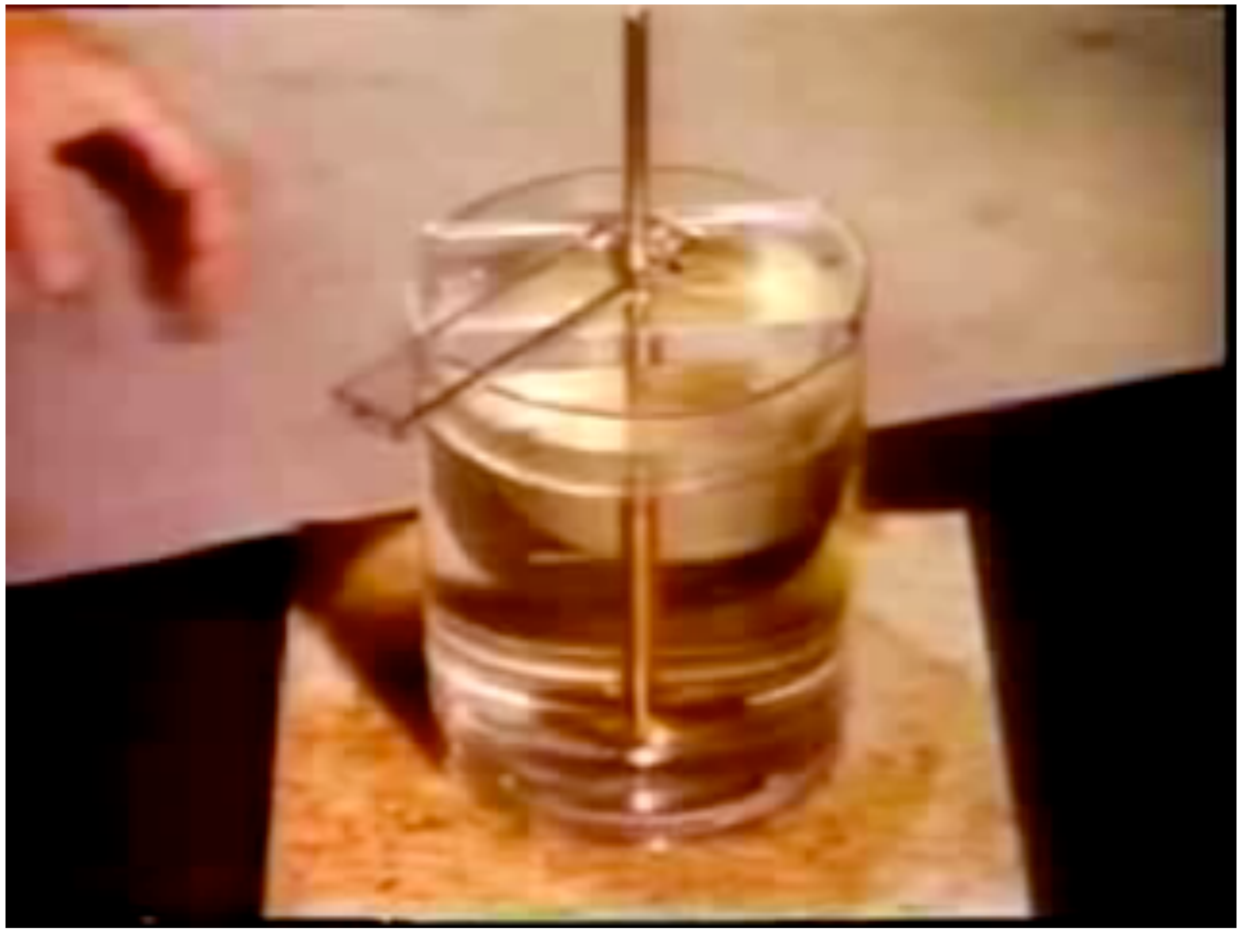
Dimensional estimate of intertial and viscous forces characterizes motion in fluid as laminar or turbulent



Life at Low Reynolds Number, G. I. Taylor 1972

How cells swim

No coasting or turbulence in laminar flow
E. coli evolved flagella to break symmetry



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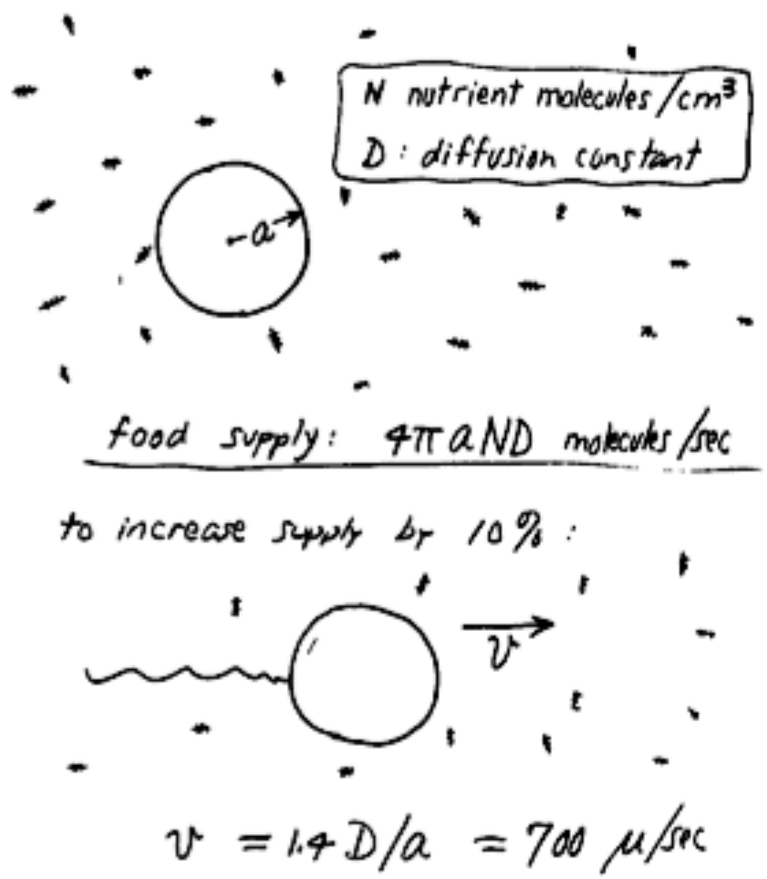
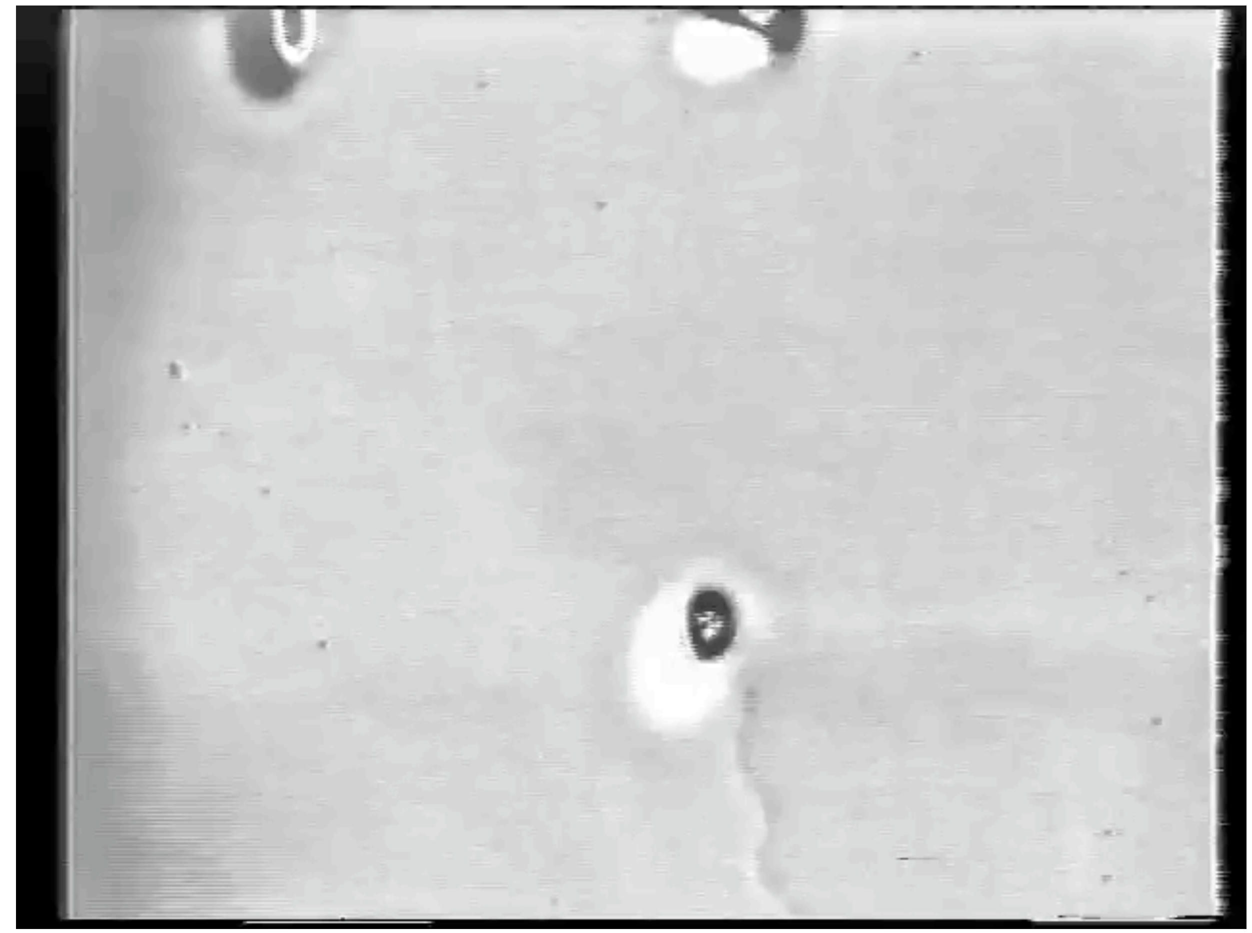




Why cells swim

Cells don't "outswim" diffusion; Run and tumble mechanism is a search for "greener pastures"

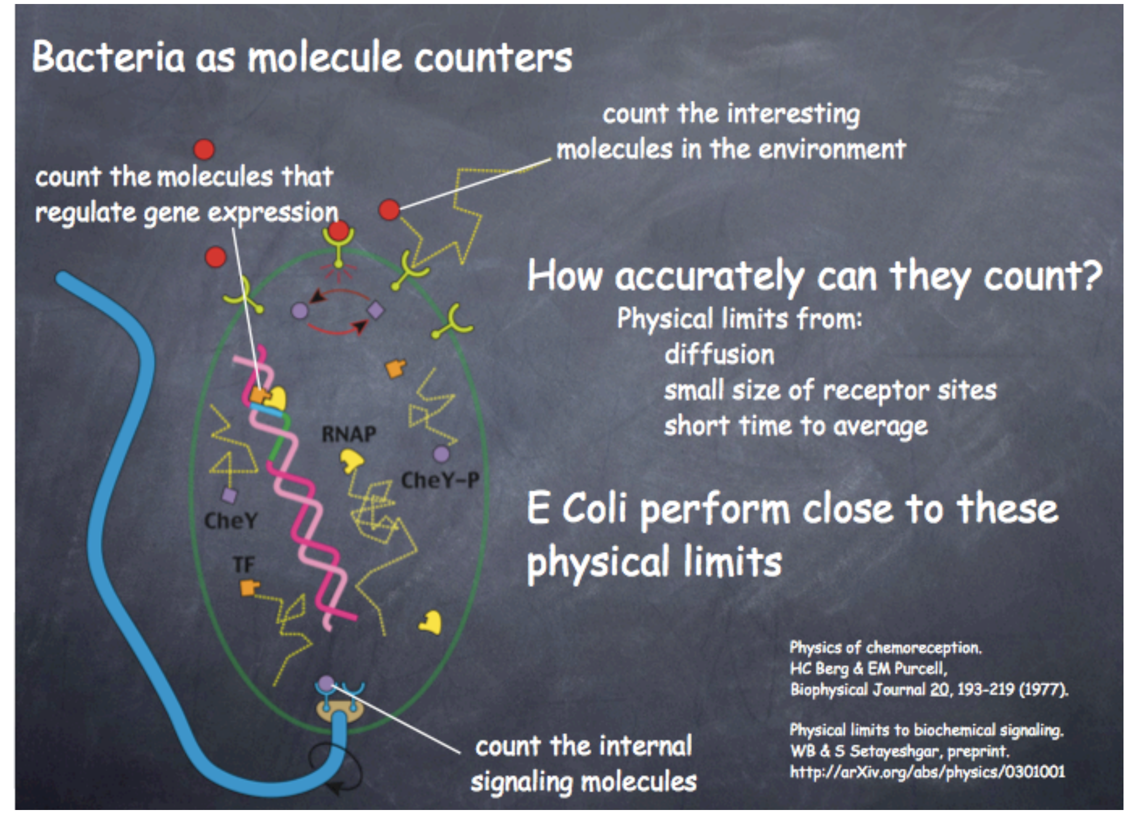
Physical limits to biochemical signaling

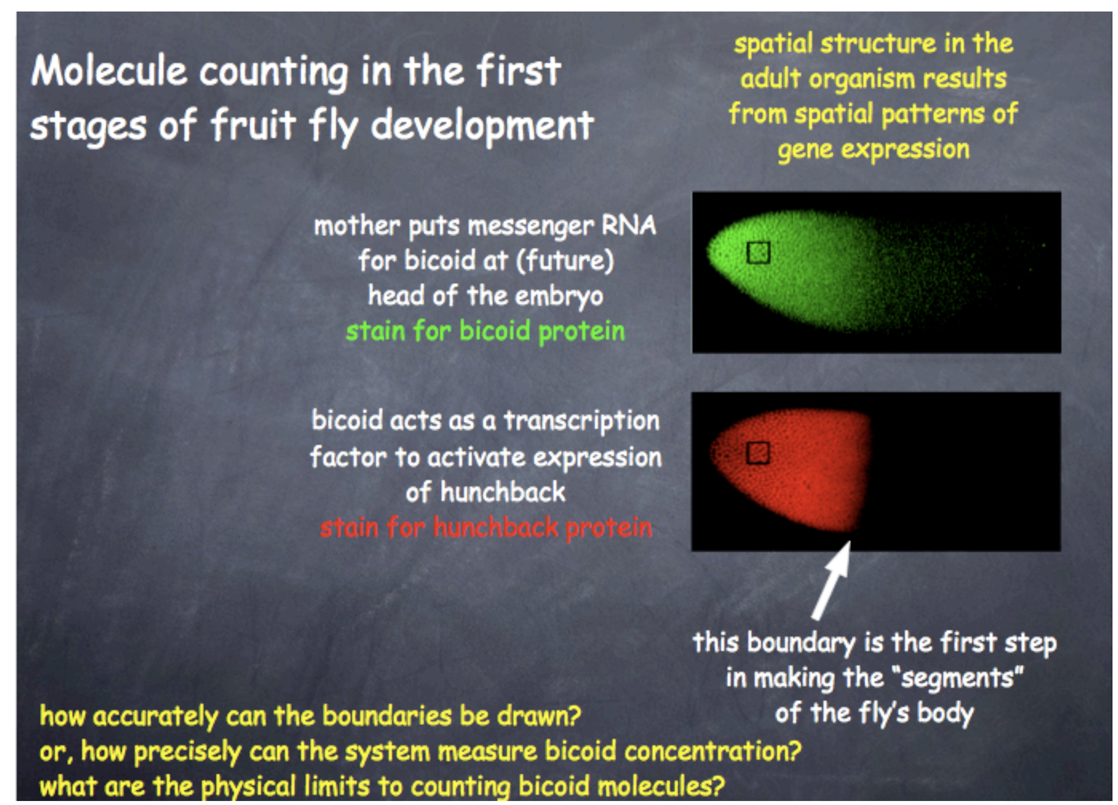
William Bialek* and Sima Setayeshgar†

Joseph Henry Laboratories of Physics and Lewis-Sigler Institute for Integrative Genomics, Princeton University, Princeton, NJ 08544

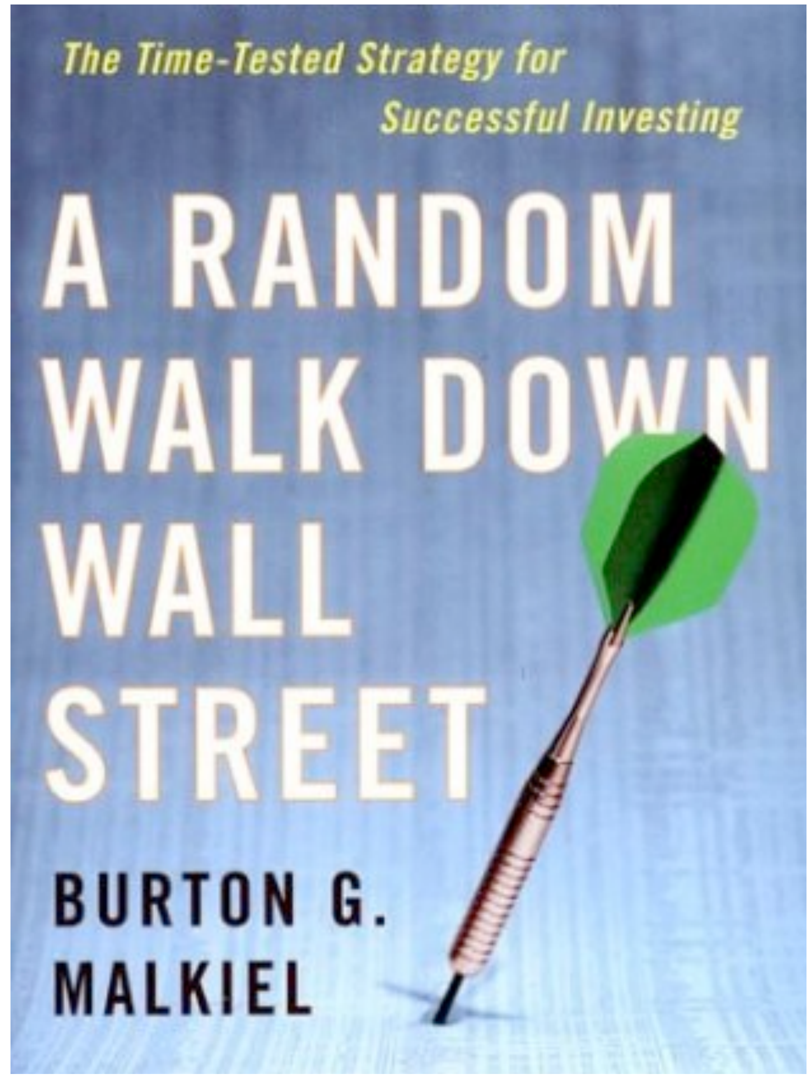
The analysis of bacterial chemotaxis by Berg and Purcell (6) provided a clear intuitive picture of the noise in "measuring" chemical concentrations. Their argument was that if we have a sensor with linear dimensions a, we expect to count an average of $\bar{N} \sim \bar{c}a^3$ molecules when the mean concentration is $\bar{c}$. Each such measurement, however, is associated with a noise $\delta N_1 \sim \sqrt{\bar{N}}$. A volume with linear dimension a can be cleared by diffusion in a time $\tau_D \sim a^2/D$, so if we are willing to integrate over a time τ we can make $N_{\mathrm{meas}} \sim \tau/\tau_D$ independent measurements, reducing the noise in our estimate of N by a factor of $\sqrt{N_{\mathrm{meas}}}$. The result is that our fractional accuracy in measuring N, and hence in measuring the concentration c itself, is given by

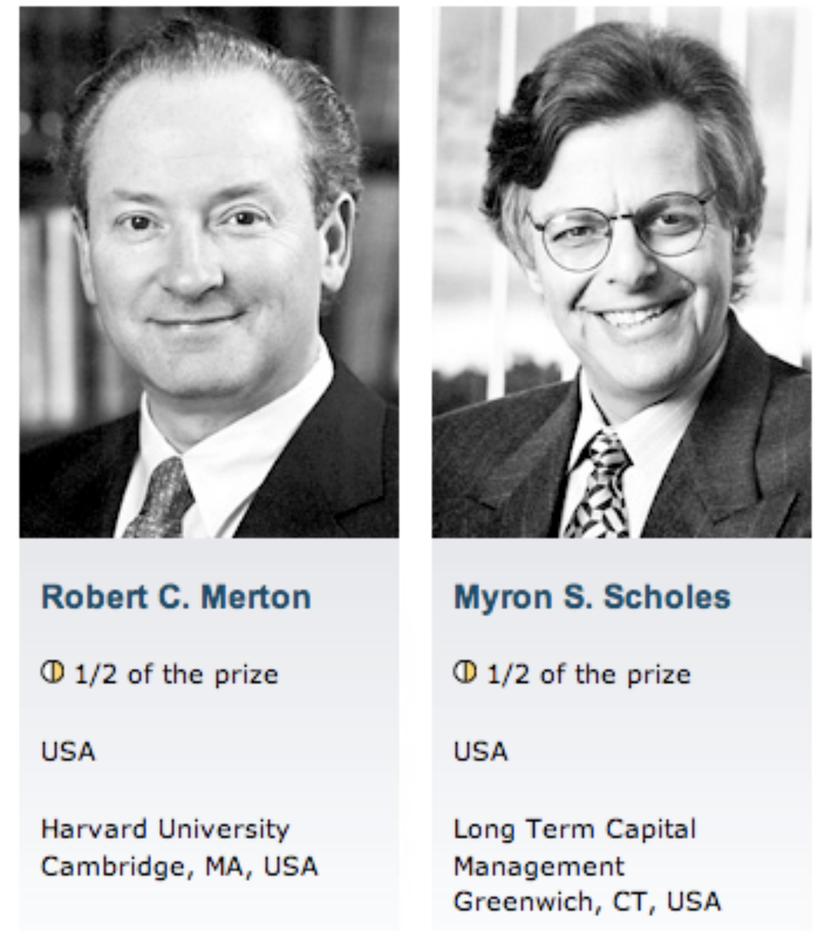
$$\frac{\delta c}{\bar{c}} = \frac{\delta N}{\bar{N}} = \frac{1}{\sqrt{\bar{N}\, N_{\mathrm{meas}}}} = \frac{1}{\sqrt{Da\bar{c}\tau}}. \qquad [1]$$





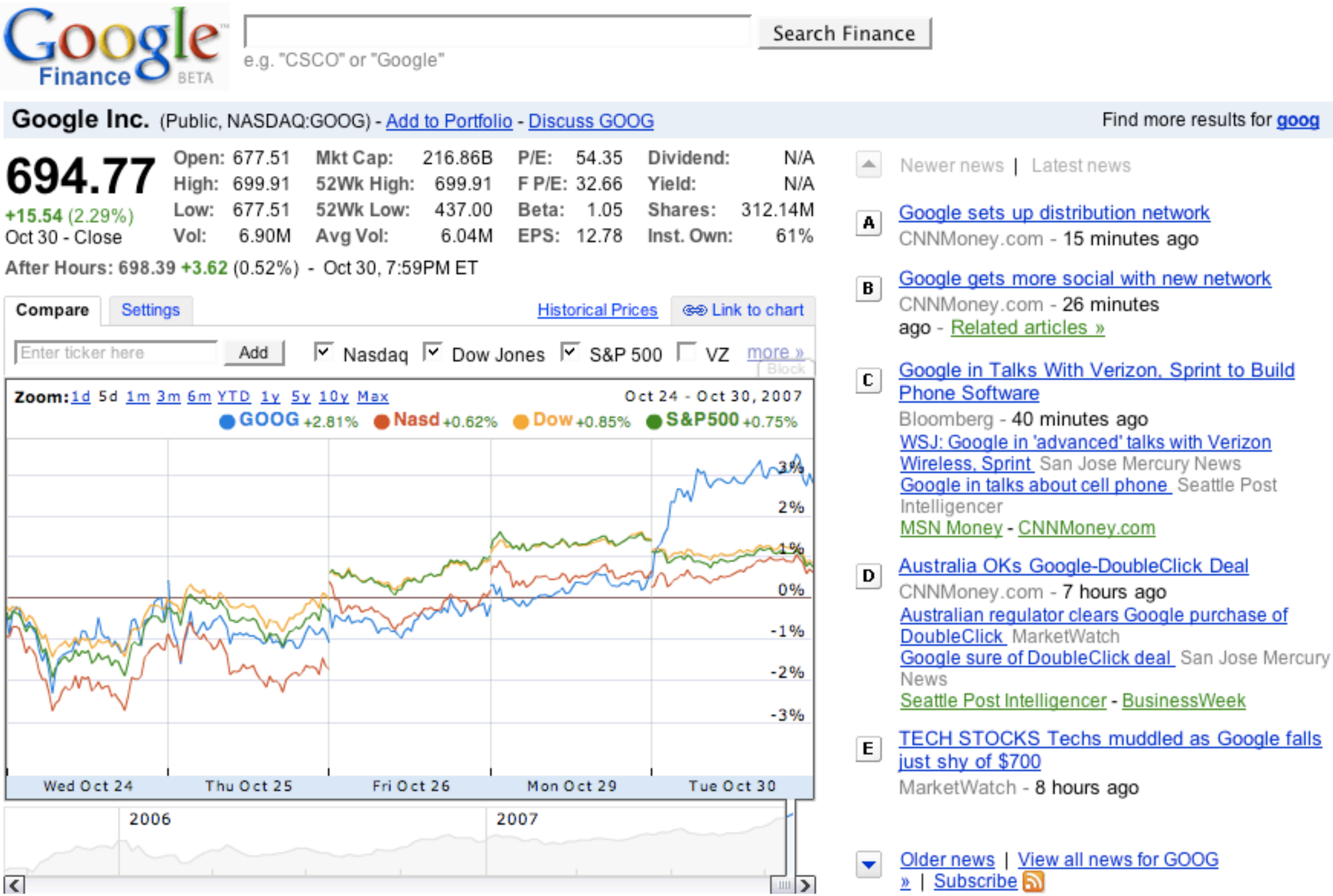
http://www.princeton.edu/~wbialek/wbialek.html





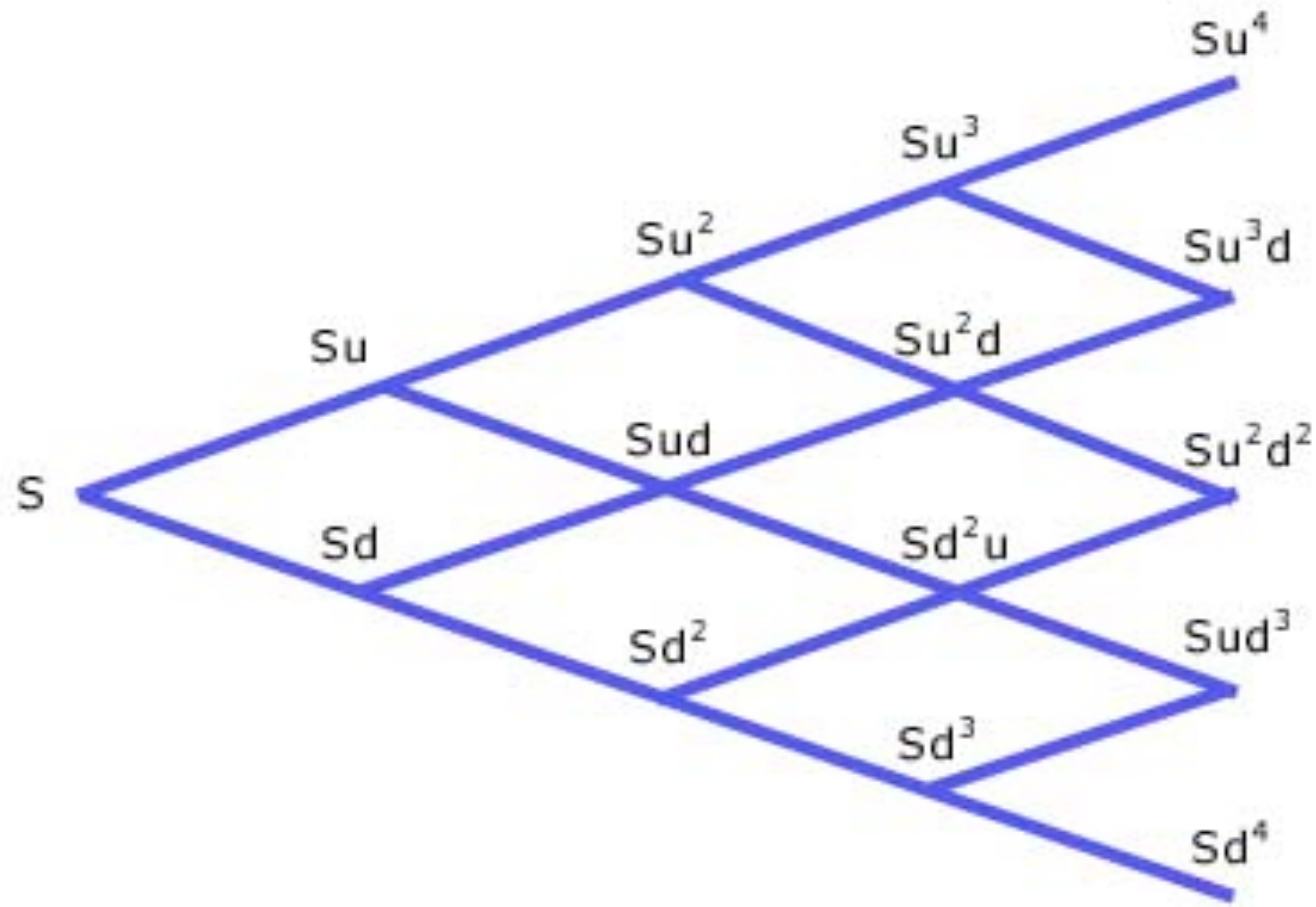
Return rates as a random walk

Bachelier's thesis 1900
Black, Scholes, Merton 1973
Cox, Ross, Rubenstein 1979



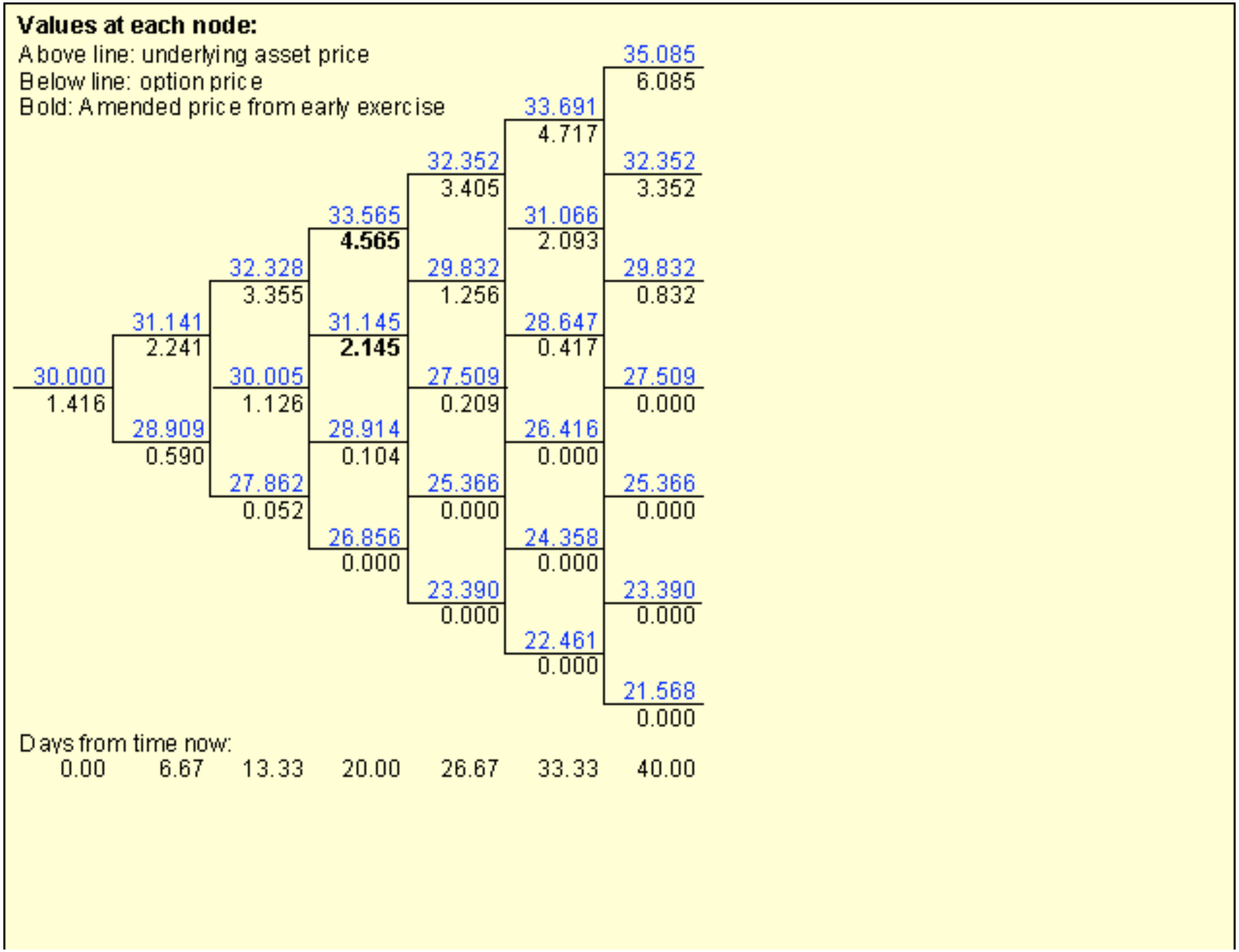
Options pricing

What is the fair price for a contract to buy or sell a stock at a fixed price on a future date?



Binomial options pricing

Geometric random walk:
Price of stock increases or decreases by *multiplicative* factor at each time step with *drift*



Binomial options pricing

Use geometric random walk to calculate final prices
Work backwards to calculate fair option price at each time

http://www.hoadley.net/options/binomialtree.aspx

Conclusions

- Simple mathematics, e.g. probability and statistics, describe complex processes

- Individual events are random or unpredictable, but collective behavior can be quantified reliably

- Mathematical framework of random walks give united description of diverse systems, e.g. molecular biology, financial markets, wireless networking, Google's PageRank

- Dimensional analysis useful for basic understanding of physical phenomena

Acknowledgments

- Images stolen from Wikipedia unless otherwise noted
- *Biological Physics*, Philip C. Nelson
- Advisor: Chris H. Wiggins

Thank you.
Questions?