Franklin (done by Rayleigh):

teaspoon = $5\ cm^3 = 5 \times (10^{-2})^3 = 5 \times 10^{-6} m^3$

$\frac{1}{2}$ acre = $2000\ m^2$

to get a length: $\frac{\text{Volume}}{\text{area}} = \frac{5 \times 10^{-6} m^3}{2 \times 10^3 m^2}$

Dimensional analysis

$= \underbrace{2.5 \times 10^{-9} m}_{\text{nanometers}}$

(can do w/o scientific notation too.)

# Brownian motion

steps of length $\ell$ every $\Delta t$ in time.

say equal chance of moving left/right at each step.

(indp. coin flips)

explain notation (brackets, subscript)

**1 step:**

| | $x$ | $x^2$ |
|---|---|---|
| + | $+\ell$ | $\ell^2$ |
| − | $-\ell$ | $\ell^2$ |

averages:

$$\langle x_1 \rangle = \frac{(+\ell)+(-\ell)}{2} = 0$$

$$\langle x_1^2 \rangle = \frac{(+\ell)^2+(+\ell)^2}{2} = \ell^2$$

**2 steps:**

| | | $x$ | $x^2$ |
|---|---|---|---|
| 1 | ++ | $+2\ell$ | $4\ell^2$ |
| 2 | +− | 0 | 0 |
| | −+ | 0 | 0 |
| 1 | −− | $-2\ell$ | $4\ell^2$ |

averages:

$$\langle x_2 \rangle = \frac{(+2\ell)+0+0+(-2\ell)}{4} = 0$$

$$\langle x_2^2 \rangle = \frac{4\ell^2+0+0+4\ell^2}{4} = 2\ell^2$$

**3 steps:**

| | | $x$ | $x^2$ |
|---|---|---|---|
| 1 | +++ | $+3\ell$ | $9\ell^2$ |
| 3 | ++− | $+\ell$ | $\ell^2$ |
| | +−+ | $+\ell$ | $\ell^2$ |
| | −++ | $+\ell$ | $\ell^2$ |
| 3 | −−+ | $-\ell$ | $\ell^2$ |
| | −+− | $-\ell$ | $\ell^2$ |
| | +−− | $-\ell$ | $\ell^2$ |
| 1 | −−− | $+3\ell$ | $9\ell^2$ |

$$\langle x_3 \rangle = \frac{1\cdot(+3\ell)+3(+\ell)+3(-\ell)+1(-3\ell)}{8} = 0$$

$$\langle x_3^2 \rangle = \frac{1\cdot(9\ell^2)+3(\ell^2)+3(\ell^2)+1(9\ell^2)}{8} = 3\ell^2$$

First, note that we have a pattern:

$$\langle x_n^2 \rangle = n\ell^2 \qquad (\langle x_n \rangle = 0)$$

characterizes brownian motion. grow as sqrt of # steps

explore implications further in moment.

now can we count a bit more smartly?

① total # outcomes is $2^N$ for $N$ steps.

② looking for way to count # heads in $N$ flips, order is irrelevant.

alt:

$$\underbrace{\underline{N}\ \ \underline{N-1}\ \ \underline{N-2}\ \ \_\ \_\ \_}_{K \text{ heads}} \;\Big|\; \underbrace{\_\ \_\ \ \_\ \_}$$

$N$ flips; $N$ choices for 1st, $N-1$ ...

but "first" is irrelevant; divide out meaningless rearrangements ($K!$ of them)

fancy notation for "choose" or "binomial" coefficients: $${}_N C_k = \frac{N!}{(N-k)!\,k!}$$

confirm this with list.

can we take averages more intelligently?

yes, multiply outcome by relative frequency, or <u>probability</u> of occurrence:

$$\langle x_3^2 \rangle = \frac{1}{8}(9\ell^2) + \frac{3}{8}(\ell^2) + \frac{3}{8}(\ell^2) + \frac{1}{8}(9\ell^2) = 3\ell^2$$

$$\langle x_3 \rangle = \frac{1}{8}(-3\ell) + \frac{3}{8}(-\ell) + \frac{3}{8}(+\ell) + \frac{1}{8}(3\ell) = 0$$

fancy notation: $\langle f(x) \rangle = p(x_1)\,f(x_1) + p(x_2)\,f(x_2) + \ldots$

expected value of $f$.

now we can generalize our results:

to get $k$ heads in $N$ flips: ${}_N C_k$

total possible outcomes: $2^N$

probability: $\dfrac{\text{\# ways to get want}}{\text{total \# outcomes}} = \dfrac{{}_N C_k}{2^N}$

can also visualize by starting w/ concentrated source, watch particles diffuse:

$$1$$

1 step: $\frac{1}{2} \quad \frac{1}{2}$

2 steps: $\frac{1}{4} \quad \underbrace{\frac{1}{4}+\frac{1}{4}}_{\frac{2}{4}} \quad \frac{1}{4}$

3 steps: $\frac{1}{8} \quad \underbrace{\frac{1}{8}+\frac{1}{4}}_{\frac{3}{8}} \quad \underbrace{\frac{1}{4}+\frac{1}{8}}_{\frac{3}{8}} \quad \frac{1}{8}$

Pascal's triangle divided by $2^N$!

etc

limit as $N \to \infty$, get Bell/Normal/Gaussian!

[MATLAB DEMOS]

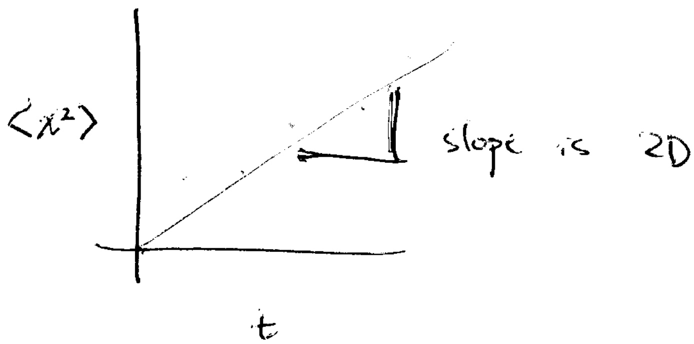
probability of landing at any particular spot very small;

but $\underbrace{p(\text{+NL})}_{\sim 10^{-3000}} \gg \underbrace{p(C)}_{\sim 10^{-3}}$! observe <u>rare</u> and <u>large</u> deviations (two problems cancel) can't see individual steps, just watch time

$$\# \text{ steps in time } t = \frac{t}{\Delta t}$$

$$\langle x_n^2 \rangle = n\ell^2 = \frac{1}{2}\frac{2t}{\Delta t}\ell^2 = 2\underbrace{\left(\frac{\ell^2}{2\Delta t}\right)}_{\text{diffusion const, } D.} t$$

Characterizes physical motion and is experimentally measurable




relate ~~macroscopic~~ micro and ~~microscopic~~ nano

diffusion is ~~small~~ useful ~~for~~ for small scales, terrible for large scales

$D_{\text{water, room temp}} \approx 1\ \mu m^2/\text{min}$ ← effective transport at level of cell.

minimal physics:

$F = ma$ dimensional analysis $[F] = \frac{M \cdot L}{T^2}$

drag force is proportional to velocity (wind). $F = -bv$

$$[b] = \frac{ML}{T^2} \cdot \frac{T}{L} = \frac{M}{T}$$

↑ drag coeff, $\zeta$ to be fancy

~~Work~~ or energy: Force x dist (~~weight lifting~~)

So what do we have at disposal:

$$[D] = \frac{L^2}{T} \qquad [b] = \frac{M}{T}$$

$$\rightarrow [Db] = \underbrace{\frac{M \cdot L}{T^2}}_{\text{force}} \cdot \underbrace{L}_{\text{dist}} = \text{Energy} \qquad \text{(e.g. weightlifting)}$$

Where does this energy come from?

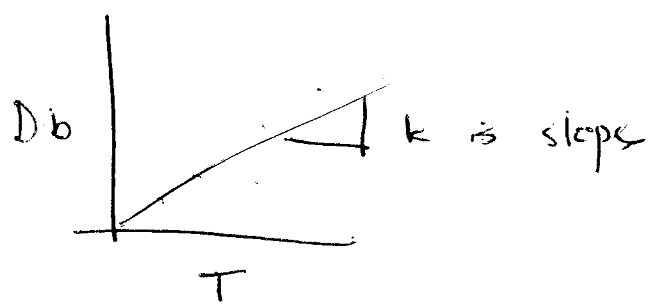
Temperature: average energy of motion

So somehow

$$Db = kT$$

↑ converts units, called Boltzmann's const.

now have 3 macroscopically measureable quantities




can use these for estimates of molecular mass and step size!

$D(T)$ complicated, $b(T)$ Complicated

→ but $Db$ simple!

called "fluctuation-dissipation" thrm.

# Betting & Expected Values

play game: flip two fair coins

if 1 or 2 heads, you get \$2

if none, get nothing

what's a "fair" price?

$$\langle w \rangle = \frac{3}{4}(2-\$) + \frac{1}{4}(0-\$)$$

net winnings for each outcome

fair means: you don't win/lose anything on average

$$0 = \frac{3}{4}(2-\$) + \frac{1}{4}(0-\$)$$

$$\$ = \frac{3}{2} \longrightarrow \$1.50$$

# Stock Options

history: chicago farmers futures, had to buy at exp.

options: have option to exercise contract

so, what is fair value of an options contract?

rules of game:
- own $h$ shares of stock priced $S$
- write a contract $C$, for right to buy at $E$.
- want value of $C$
- want your holdings to grow at standard interest rate $r$ either way

"fair" means get $(1+r)$ *, not zero.

binomial model:

$$\begin{tikzcd} & S_u = (1+u)S \\ S \arrow[ur, "p"] \arrow[dr, "1-p"'] & \\ & S_d = (1+d)S \end{tikzcd}$$

payoff (out of your pocket)

$C_u = \max(S_u - E, 0)$

↳ write piecewise fcn for payout!

$C_d = \max(S_d - E, 0)$

* note: $u$, $d$, and $r$ determine $p$!

value now $hS - C \xrightarrow[\text{if \$ in bank}]{}$ ① $\underbrace{(1+r)(hS-c)}$

$\downarrow$ stock moves

stock up: ② $hS_u - C_u$

stock down: ③ $hS_d - C_d$ $\Big\} \longrightarrow$ want all of these equal

②=③ $\Longrightarrow h = \dfrac{C_u - C_d}{S_u - S_d}$

①=② $\Longrightarrow (1+r)(hS - c) = hS_u - C_u$

$$C = hS - \frac{hS_u - C_u}{1+r}$$

regroup, sub in for $h$:

$$C = \frac{[pC_u + (1-p)C_d]}{1+r} \quad \text{where } p \equiv \frac{r-d}{u-d}$$

↑ so market data determine $p$.

* geometric brownian motion w/ drift set by interest rate *

to get fair price, make expected value of portfolio same value as "sure thing" interest from bank/bond.

example: $S=100$, $E=95$ $u=0.10$, $d=-0.10$, $r=0.05$

$\rightarrow S_u = 110$ $C_u = 15$

$S_d = 90$ $C_d = 0$

$$h = \frac{15-0}{110-90} = \frac{3}{4} \qquad p = \frac{0.05+0.10}{0.10+0.10} = \frac{3}{4}$$

own $\frac{3}{4}$ of stock

$\frac{3}{4}$ prob up
$\frac{1}{4}$ prob. down

$$C = \frac{\frac{3}{4}\cdot 15 + \frac{1}{4}\cdot 0}{1.05} = 10.71$$

Check: value now:

$$\frac{3}{4}\cdot 100 - 10.71 = 64.29$$

if invested at riskless 5%:

$$1.05 \cdot 64.29 = 67.50$$

if goes up:

$$\frac{3}{4}\cdot 110 - 15 = 67.50$$

if goes down:

$$\frac{3}{4}\cdot 90 = 67.50$$

all equal!

http://www.sjsu.edu/faculty/watkins/binomial.htm