1 the gaussian approximation to the binomial

we start with the probability of ending up j steps from the origin when taking a total of N steps, given by

$$P_j = \frac{N!}{2^N \left(\frac{N+j}{2}\right)! \left(\frac{N-j}{2}\right)!} \tag{1}$$

taking the logarithm of both sides, we have

$$\ln P_j = \ln N! - N \ln 2 - \ln \left(\frac{N+j}{2}\right)! - \ln \left(\frac{N-j}{2}\right)! \tag{2}$$

now we apply stirling's approximation, which reads

$$\ln N! \approx N \ln N - N + \frac{1}{2} \ln 2\pi N \tag{3}$$

for large N. this gives

$$\begin{aligned} \ln P_j \approx & \left[N \ln N - N + \frac{1}{2} \ln 2\pi N\right] - N \ln 2 - \\ & \left[\left(\frac{N+j}{2}\right) \ln \left(\frac{N+j}{2}\right) - \left(\frac{N+j}{2}\right) + \frac{1}{2} \ln 2\pi \left(\frac{N+j}{2}\right)\right] - \\ & \left[\left(\frac{N-j}{2}\right) \ln \left(\frac{N-j}{2}\right) - \left(\frac{N-j}{2}\right) + \frac{1}{2} \ln 2\pi \left(\frac{N-j}{2}\right)\right] \end{aligned}$$

the second term in each of the square brackets cancel each other. regrouping the first term in each of the square brackets together,

$$\begin{aligned} \ln P_j \approx & \left[N \ln N - \left(\frac{N+j}{2}\right) \ln \left(\frac{N+j}{2}\right) - \left(\frac{N-j}{2}\right) \ln \left(\frac{N-j}{2}\right)\right] && (4) \\ & + \frac{1}{2} \ln 2\pi N - \frac{1}{2} \ln 2\pi \left(\frac{N+j}{2}\right) - \frac{1}{2} \ln 2\pi \left(\frac{N-j}{2}\right) - N \ln 2 && (5) \end{aligned}$$

looking at only the terms in square brackets and rearranging a bit,

$$\begin{aligned} [\ \ ] &= N \ln N - \frac{N}{2} \ln \left(\frac{N+j}{2}\right)\left(\frac{N-j}{2}\right) - \frac{j}{2} \ln \left(\frac{N+j}{2}\right) + \frac{j}{2} \ln \left(\frac{N-j}{2}\right) \\ &= N \ln N - \frac{N}{2} \ln \left[\frac{N^2}{4}(1 - \frac{j^2}{N^2})\right] - \frac{j}{2} \ln \left[\frac{N}{2}(1 + \frac{j}{N})\right] + \frac{j}{2} \ln \left[\frac{N}{2}(1 - \frac{j}{N})\right] \\ &= N \ln N - \frac{N}{2} \ln \frac{N^2}{4} - \frac{N}{2} \ln(1 - \frac{j^2}{N^2}) - \frac{j}{2} \ln(1 + \frac{j}{N}) + \frac{j}{2} \ln(1 - \frac{j}{N}) \end{aligned}$$

now we use the taylor expansion $\ln(1 \pm x) \approx \pm x$ for $x \ll 1$ and work to "second order in $\frac{j}{N}$":

$$\begin{aligned} [\ \ ] &\approx N \ln 2 + \frac{N}{2} \frac{j^2}{N^2} - \frac{j}{2} \frac{j}{N} - \frac{j}{2} \frac{j}{N} \\ &= N \ln 2 - \frac{j^2}{2N} \end{aligned}$$

simplifying Eqn. (4) a bit and plugging the above in gives

$$\ln P_j \approx \left[N \ln 2 - \frac{j^2}{2N}\right] - \frac{1}{2} \ln \left[\frac{2\pi}{N} \left(\frac{N+j}{2}\right) \left(\frac{N-j}{2}\right)\right] - N \ln 2 \tag{6}$$

now we approximate the second term in the same manner as before:

$$\begin{aligned} \ln P_j &\approx -\frac{j^2}{2N} - \frac{1}{2} \ln \left[\frac{\pi}{2N} N^2 (1 - \frac{j^2}{N^2})\right] \\ &\approx -\frac{j^2}{2N} - \frac{1}{2} \ln \frac{\pi N}{2} \end{aligned}$$

we exponentiate to get P_j back:

$$P_j \approx \sqrt{\frac{2}{\pi N}} e^{-\frac{j^2}{2N}} \tag{7}$$

to get $P(x) = \frac{P_j}{2L}$, we use $j = x/L$ and $L = \sqrt{2Dt/N}$:

$$\begin{aligned} P(x) &= \frac{1}{2}\sqrt{\frac{N}{2Dt}}\sqrt{\frac{2}{\pi N}} \exp(-\frac{x^2}{2N}\frac{N}{2Dt}) \quad (8) \\ &= \frac{1}{\sqrt{4\pi Dt}} \exp(-\frac{x^2}{4Dt}) \quad (9) \end{aligned}$$

which is, at last, a gaussian distribution.

lesson: the large-N limit of a "fair" binomial distribution is a gaussian distribution.